Two Facts about the Cubic Surface

Gabriel Ong

Fall 2022

Abstract

We give an exposition of two theorems relating to cubic surfaces in $\mathbb{P}^3_{\mathbb{C}}$ – the existence of 27 lines and its realization as the blowup of $\mathbb{P}^2_{\mathbb{C}}$ in six points. We discuss the connections between these two theorems.

1 Introduction

In this paper, we will give an exposition of the following two facts about the cubic surface.

Theorem. *Let $X \subseteq \mathbb{P}^3_{\mathbb{C}}$ be a smooth cubic surface. S contains exactly 27 lines.*

Theorem. *Let $X \subseteq \mathbb{P}^3_{\mathbb{C}}$ be a smooth cubic surface. S can be realized as the blowup of $\mathbb{P}^2_{\mathbb{C}}$ at six points, no three in a line and no six in a conic.*

We work over $\mathbb{C}$ and follow closely the exposition of [1], with reference to [4, 5].

2 The 27 Lines on a Cubic Surface

Let us start with the following example.

Proposition 2.1. *The surface $X = \mathcal{V}(w^3 + x^3 + y^3 + z^3) \subseteq \mathbb{P}^3_{\mathbb{C}}$ contains exactly 27 lines.*

The surface $X = \mathcal{V}(w^3 + x^3 + y^3 + z^3) \subseteq \mathbb{P}^3_{\mathbb{C}}$ is known as the Fermat cubic surface.

Proof. Recall from [2, Ex. 1.6] that any ordered tuple of 5 points are projectively equivalent in $\mathbb{P}^3_{\mathbb{C}}$. Note that a line in $\mathbb{P}^3_{\mathbb{C}}$ is determined by the intersection of two hyperplanes say $\mathcal{V}(a_0'w + a_1'x + a_2'y + a_3'z, b_0'w + b_1'x + b_2'y + b_3'z) \subseteq \mathbb{P}^3_{\mathbb{C}}$. Thus without loss of generality, we can change coordinates such that our hyperplanes are of the form $w - a_2y - a_3z$ and $x - b_2y - b_3z$ that preserves the coordinates of X.

Two lines lie on X if and only if

$$(a_2y + a_3z)^3 + (b_2y + b_3z)^3 + y^3 + z^3 = 0$$

as a polynomial in $\mathbb{C}[y,z]$. Expanding and comparing coefficients, we see that the above polynomial vanishes if and only if the following conditions hold:

$$\begin{aligned}
&a_2^3 + b_2^3 + 1 = 0 && (\dagger)\\
&a_3^3 + b_3^3 + 1 = 0 && (\dagger\dagger)\\
&a_2^2 a_3 + b_2^2 b_3 = 0 && (\ddagger)\\
&a_2 a_3^3 + b_2 b_3^2 = 0 && (\ddagger\ddagger)
\end{aligned}$$

We now want to show that at least one of a_2, a_3, b_2, b_3 is zero. Suppose to the contrary that all are nonzero. Consider the square of $(\ddagger)$ divided by $(\ddagger\ddagger)$ that gives us $a_2^3 = b_2^3$ which contradicts $(\dagger)$. So at least one of a_2, a_3, b_2, b_3 is zero.

Up to renumbering say $a_2 = 0$. With the equations above, we can compute in sucession that $b_2^3 = -1$ using $(\dagger)$, $b_3 = 0$ by $(\ddagger)$, and $a_3^3 = -1$ by $(\dagger\dagger)$. Indeed if a_2, a_3, b_2, b_3 satisfies these equations then the intersection of the hyperplanes are fully contained in the cubic surface – namely is a line in the cubic.

Let ζ be the third root of unity. Allowing permutations of the coordinates we can see that there are nine lines for each of the three families of hyperplane intersections

$$\begin{aligned}
&\mathcal{V}(x + y\zeta^i, w + z\zeta^j) && 0 \le i,j \le 2\\
&\mathcal{V}(z + x\zeta^i, w + y\zeta^j) && 0 \le i,j \le 2\\
&\mathcal{V}(z + y\zeta^i, w + x\zeta^j) && 0 \le i,j \le 2
\end{aligned}$$

which we must now show are distinct. We can check that the first collection of lines is of the form $[-s\zeta^j : -t\zeta^i : t : s], [s : t] \in \mathbb{P}^1_{\mathbb{C}}$, the second collection of lines is of the form $[-s\zeta^j : t : s : -t\zeta^i], [s : t] \in \mathbb{P}^1_{\mathbb{C}}$, and the third collection of lines are of the form $[-s\zeta^j : s : t : -t\zeta^i], [s : t] \in \mathbb{P}^1_{\mathbb{C}}$. We can see these are distinct for different choices of i, j. $\square$

Indeed one can compute the following.

Corollary 2.2. *Let X be the Fermat cubic surface. For any line $\ell \subset X$ there are ten other lines in X intersecting ℓ. Given two disjoint lines $\ell, \ell' \subset X$ there are exactly five other lines in X meeting both ℓ, ℓ'.*

We wish to genearalize this. By a parameter counting argument, a homogeneous cubic polynomial in $\mathbb{C}[w,x,y,z]$ is determined by 20 coefficients. Its vanishing locus is determined by these 20 coefficients up to scale, hence the space of cubic surfaces is $\mathbb{P}^{19}_{\mathbb{C}}$. By the hypotheses in the theorem stated in our introduction, we are considering *smooth* cubic surfaces. Here, we note that a singularity is a closed condition in the space of cubic surfaces – namely that the set of singular cubic surfaces is defined by the vanishing of the Jacobian determinant, hence is a Zariski closed subset of $\mathbb{P}^{19}_{\mathbb{C}}$. Equivalently, the space of smooth cubic surfaces is the complement of $\mathbb{P}^{19}_{\mathbb{C}}$ by this Zariski closed set and is thus some dense open subset $U \subseteq \mathbb{P}^{19}_{\mathbb{C}}$. Moreover, recall that a line in $\mathbb{P}^3_{\mathbb{C}}$ is the projectivization of a 2-plane in $\mathbb{C}^4$. Namely lines in $\mathbb{P}^3_{\mathbb{C}}$ are points of the Grassmanian $\mathrm{Gr}(2,4)$. We can thus consider the following incidence correspondence:

$$\Phi = \{(X,\ell) | \ell \subset X\} \subseteq U \times \mathrm{Gr}(2,4)$$

We note that Φ has a natural forgetful map π taking (X, ℓ) to X. The number of lines contained in X is thus the size of the fiber over π. The rest of the proof shall proceed, counterintuitively, by a topological argument. We show the following two lemmas.

Lemma 2.3. *Let Φ be the incidence correspondence as above. Φ is closed in the Zariski topology of $U \times \mathrm{Gr}(2,4)$.*

Proof. Let $(X, \ell) \in \Phi$. Up to the action of $\mathrm{PGL}_4(\mathbb{C})$ we can assume that L is given by the equation $y = z = 0$. Locally around this point $\ell \in \mathrm{Gr}(2,4)$ a line would be interpolating the points given by the rows of the following matrix:

$$\begin{bmatrix} 1 & 0 & a_2 & a_3 \\ 0 & 1 & b_2 & b_3 \end{bmatrix}$$

for $a_2, a_3, b_2, b_3 \in \mathbb{C}$. Note here that $a_2 = a_3 = b_2 = b_3 = 0$ corresponds to our line ℓ. Let f be the degree three homogeneous polynomial in 4 variables such that $\mathcal{V}(f) = X$. If $(X, \ell) \in \Phi$ then we have that the span of the two vectors contained entirely within the cubic surface

$$f(s[1:0:a_2:a_3] + t[0:1:b_2:b_3]) = 0, \forall [s:t] \in \mathbb{P}^1_{\mathbb{C}}.$$

Let c_α be the multi-indexed coefficients of f, namely α runs over all quadruples $(\alpha_0, \alpha_1, \alpha_2, \alpha_3)$ where $\sum_{i=0}^{3} \alpha_i = 3$ and $0 \le \alpha_i \le 3$. Given the above, we can write

$$\sum_{\alpha} c_\alpha s^{\alpha_0} t^{\alpha_1} (sa_2 + tb_2)^{\alpha_2} (sa_3 + tb_3)^{\alpha_3} = 0, \forall [s:t] \in \mathbb{P}^1_{\mathbb{C}}.$$

We can now expand and reorder terms writing the above polynomial as a homogeneous degree three polynomial in variables s, t and coefficients $a_0, \dots, a_3; b_0, \dots, b_3; c_\alpha$. Let us write these coefficient polynomials as $F(a, b; c_\alpha)$. So we have

$$\sum_{k=0}^{3} s^k t^{3-k} F_k(a, b; c_\alpha) = 0, \forall [s:t] \in \mathbb{P}^1_{\mathbb{C}}.$$

But since this has to vanish for all $[s:t] \in \mathbb{P}^1_{\mathbb{C}}$, the $F_k(a, b; c_\alpha)$ must vanish simultaneously, but this is the vanishing of a collection of bihomogeneous polynomials in the variables $(a, b; c_\alpha)$ of bidegree $(3, 1)$ on the product $U \times \mathrm{Gr}(2,4)$ showing our claim. $\square$

Now we want to show that the equations $F_0, \dots, F_3$ that these four equations determine the values $(a_2, a_3, b_2, b_3) \in \mathbb{C}^4$ locally around the origin in the analytic topology near c_α. To do this, we will need the complex implicit function theorem. Let us recall the statement. Let $f_j(w, z)$ for $1 \le j \le m$ be functions in varibles (w, z) and analytic in a neighbhorhood $(w_0, z_0) \in \mathbb{C}^n \times \mathbb{C}^m$ such that $f_j(w_0, z_0) = 0$ for all $1 \le j \le m$ have a uniquely determined analytic solution $\mathbb{C}^m \to \mathbb{C}^n$ such that $w(z_0) = w_0$. In other words, the theorem tells us that if the Jacobian determinant is nonvanishing the number of lines is locally constant in a small neighbhorhood of a given point of the incidence correspondence. It thus suffices to show that the Jacobian matrix $\frac{\partial(F_0, F_1, F_2, F_3)}{\partial(a_2, a_3, b_2, b_3)}$ is invertible at $a_2 = a_3 = b_2 = b_3 = 0$.

Lemma 2.4. *Let Φ be the incidence correspondence as above. In the induced Euclidean (analytic) topology, Φ is locally the graph of a continuously differentiable function $U \to \mathrm{Gr}(2,4)$.*

Proof. The implicit funciton theorem tells us that Φ is locally a graph if and only if the jacobian of the functions F is nonvanishing. Let us compute the Jacobian matrix at $a_2 = a_3 = b_2 = b_3 = 0$ in the variables a_2, a_3, b_2, b_3. The first column consists is of the form $(\partial_{a_2} F_0, \partial_{a_3} F_0, \partial_{b_2} F_0, \partial_{b_3} F_0)$ and

$$\begin{aligned}\frac{\partial}{\partial a_2}\left(\sum_{k=0}^{3} s^k t^{3-k} F_k\right)|_{a_2=a_3=b_2=b_3=0} &= \frac{\partial}{\partial a_2} f(s,t,sa_2+tb_2,sa_3+tb_3) \\ &= s\frac{\partial f}{\partial y}(s,t,0,0)\end{aligned}$$

recalling that f is the homogeneous cubic in $\mathbb{C}[w,x,y,z]$ whose vanishing locus is X. Repeating this computation of partial derivatives we can see that

$$J = \begin{bmatrix} | & | & | & | \\ s\frac{\partial f}{\partial y}(s,t,0,0) & s\frac{\partial f}{\partial z}(s,t,0,0) & t\frac{\partial f}{\partial y}(s,t,0,0) & t\frac{\partial f}{\partial z}(s,t,0,0) \\ | & | & | & | \end{bmatrix}$$

which is singular if and only if there is some $(\lambda, \mu, \lambda', \mu')$ such that the columns are linearly dependent. Namely

$$(\lambda s + \mu t)\frac{\partial f}{\partial y}(s,t,0,0) + (\lambda' s + \mu' t)\frac{\partial f}{\partial z}(s,t,0,0) = 0.$$

But recall from the fundamental theorem of algebra that a homogeneous polynomial in two variables s, t always decomposes as linear factors. So that means that $\frac{\partial f}{\partial y}(s,t,0,0)$ and $\frac{\partial f}{\partial z}(s,t,0,0)$ must have a common linear factor. So there must be a point p on the line $s[1:0:0:0]+t[0:1:0:0]$ for all $[s:t] \in \mathbb{P}^1_{\mathbb{C}}$ such that $\frac{\partial f}{\partial y}(p) = \frac{\partial f}{\partial z}(p) = 0$. So all partial derivatives of f vanish at p indicating X is singular at p, a contradiction as X is smooth by hypothesis. $\square$

This allows us to prove our theorem.

Proof of Theorem. Let $X \in U$ be a smooth cubic surface. Let $\ell \in \mathbb{P}^3_{\mathbb{C}}$ be an arbitrary line. We consider two cases.

Suppose $\ell \subset X$. **Lemma 2.4** implies that there is an open neighbhorhood of (X, ℓ) which we denote $V_\ell \times W_\ell \subseteq U \times \mathrm{Gr}(2,4)$ on which the incidence correspondence Φ is the graph of a continuously differentiable function. In particular, any cubic in V_ℓ contains exactly one line in W_ℓ.

Suppose $\ell \not\subset X$. There is an open neighborhood of (X, ℓ) which we denote $V_\ell \times W_\ell \subseteq U \times \mathrm{Gr}(2,4)$ such that no cubic in V_ℓ contains any line since the incidence correspondence Φ is closed by **Lemma 2.3**.

Recalling that X is fixed we let ℓ vary. Since $\mathrm{Gr}(2,4)$ is a projective variety, it is compact, and there are thus finitely many sets W_ℓ that form a finite open cover of $\mathrm{Gr}(2,4)$. Let for such a collection of W_ℓ, consider their intersection W and the corresponding intersection of V_ℓs which we denote $V \subseteq U$. This gives us $V \times W \subseteq U \times \mathrm{Gr}(2,4)$ on which the number of lines on a cubic surface in V is locally constant, the number of lines on the fixed cubic surface X. But since X was fixed arbitrarily, the number of lines on a cubic surface is a locally constant function on U.

To show that this is globally constant, it suffices to show that U is connected as a topological space. We do so by a dimension counting argument. We have already established that U is the complement of $\mathbb{P}^{19}_{\mathbb{C}}$ by a Zariski closed subset. A computation of the singular locus of cubics surfaces is defined by the common vanishing of six polynomials of degree 32 [3]. This has complex codimension at least one and hence its complement in $\mathbb{P}^{19}_{\mathbb{C}}$ remains connected. Thus the number of lines is globally constant on U.

Computing partial derivatives, one can verify that the Fermat surface is smooth and is hence contained in U. Moreover it has 27 lines as shown in **Proposition 2.1**. Thus by the number of lines being constant on U, a smooth cubic surface has 27 lines proving our theorem. $\square$

3 Birational Geometry of the Cubic Surface

We seek to give an outline of the proof of the second theorem. We must first show the following lemma.

Lemma 3.1. *Let $X \subseteq \mathbb{P}^3_{\mathbb{C}}$ be a smooth cubic surface. S is birational to $\mathbb{P}^2_{\mathbb{C}}$.*

We will describe the maps but not give an explicit coordinatewise descriptions of the maps.

Proof. From **Corollary 2.2**, there exist two disjoint lines ℓ, ℓ' contained in X. We will show that X is birational to $\ell \times \ell' \cong \mathbb{P}^1_{\mathbb{C}} \times \mathbb{P}^1_{\mathbb{C}}$ which is birational to $\mathbb{P}^2_{\mathbb{C}}$ with our statement following from the transitivity of birational maps when restricting to appropriate dense open subsets.

To see a rational map $X \to \ell \times \ell'$ we consider that for every point p on $X \setminus \ell \cup \ell'$ there is a unique line L through p intersecting ℓ, ℓ'. Take the rational map $p \mapsto (L \cap \ell, L \cap \ell')$ which is regular on the dense open set $X \setminus \ell \cup \ell'$.

To see a rational map $\ell \times \ell' \to X$ for any pair of points (p, p') we consider a line L in $\mathbb{P}^3_{\mathbb{C}}$ interpolating both p and p' and map it to the third point of intersection of L with X, namely $(p, p') \mapsto X \cap L$. This is regular on the set where $L \not\subseteq X$. $\square$

We can now give a proof of our second theorem.

Proof of Theorem. We will show that the cubic surface is the blowup of $\mathbb{P}^1_{\mathbb{C}} \times \mathbb{P}^1_{\mathbb{C}}$ at five points. The fact that $\mathbb{P}^1_{\mathbb{C}} \times \mathbb{P}^1_{\mathbb{C}}$ blown up at five points is birational to the blowup of $\mathbb{P}^2_{\mathbb{C}}$ at six points follows easily from the well-known fact that $\mathbb{P}^1_{\mathbb{C}} \times \mathbb{P}^1_{\mathbb{C}}$ blown up at a point is birational to $\mathbb{P}^2_{\mathbb{C}}$ blown up at two points.

Consider the map $\psi : X \to \ell \times \ell' \cong \mathbb{P}^1_{\mathbb{C}} \times \mathbb{P}^1_{\mathbb{C}}$ as above. We want to show that this map ψ is in fact a morphism. Suppose $p \in X \setminus \ell$ and let $H \subseteq \mathbb{P}^3_{\mathbb{C}}$ be the unique plane in $\mathbb{P}^3_{\mathbb{C}}$ containing ℓ and p, and $H' \subseteq \mathbb{P}^3_{\mathbb{C}}$ be the unique plane in $\mathbb{P}^3_{\mathbb{C}}$ containing ℓ' and p. Let $\psi : p \mapsto (H \cap \ell', H' \cap \ell)$. We can extend this to a morphism for points p on ℓ or ℓ' by taking

H, H', respectively, to be the tangent plane to X at p T_pX. This extends to a well-defined morphism $X \to \mathbb{P}^1_{\mathbb{C}} \times \mathbb{P}^1_{\mathbb{C}}$.

We now consider the locus where the inverse map $\mathbb{P}^1_{\mathbb{C}} \times \mathbb{P}^1_{\mathbb{C}}$ is not well-defined. Per the proof of **Lemma 3.1**, the inverse map is not well-defined if and only if the line L interpolating $(p, p') \in \ell \times \ell'$ is fully contained in X. Here recalling **Corollray 2.2**, these are the five lines intersecting both ℓ, ℓ'. In this case the whole line L will be mapped to (p, p') by ψ and that X is locally the blowup of at this point. So X is the blowup of $\mathbb{P}^1_{\mathbb{C}} \times \mathbb{P}^1_{\mathbb{C}}$ in 5 points, as desired. □

4 Exceptional Divisors as Lines and Other Remarks

We now want to discuss the connections between the first and second theorem we have just proven. In particular, how are the 27 lines on a cubic surface realized as the blowup of the six points of $\mathbb{P}^2_{\mathbb{C}}$. The 27 lines are realized as follows:

- The exceptional divisors of the blowup of the 6 points.
- The proper transforms of each line interpolating the two general points, of which there are $\binom{6}{2} = 15$.
- The proper transforms of each conic interpolating five general points, of which there are $\binom{6}{5} = 6$.

We see that

$$6 + 15 + 6 = 27$$

which are the 27 lines on the cubic surface. Through this theorem we can also deduce some aspects of the incidence geometry of the lines on the cubic surface. In particular that each line in the cubic surface meets

- If the line arises as the exceptional divisor of the blowup of a point say p_i, then the line lies intersects each of the strict transforms of the five lines $\overline{p_i p_j}$ for $i, j \in \{1, \ldots, 6\}$ distinct and each of the strict transforms of the five conics passing through p_i in the plane.
- If the line arises as the strict transform of a line through two of the blown up points $\overline{p_i p_j}$ then this line meets the exceptional divisors corresponding to p_i and p_j, the $\binom{4}{2} = 6$ lines interpolating two of the four remaining points $\{1, \ldots, 6\} \setminus \{i, j\}$, the two conics interpolating the two points p_i and p_j.
- If the line arises as the strict transform of a conic through five of the blown up points then it meets the five exceptional divisors corresponding to each of the points the conic interpolates as well as the strict transform of five lines through each of the five points on the conic and the sixth remaining line.

References

[1] Andreas Gathmann. *Chapter 11: The 27 Lines on a Smooth Cubic Surface*. Online. 2021/22. URL: https://www.mathematik.uni-kl.de/~gathmann/class/alggeom-2014/alggeom-2014-c11.pdf.

[2] Joe Harris. *Algebraic Geometry: A First Course*. Springer, 1992.

[3] Hanieh Keneshlou. "Cubic Surfaces on the ingular Locus of the Eckhart Hypersurface". In: *Le Matematiche* LXXV.2 (2020).

[4] Miles Reid. *Chapters on Algebraic Surfaces*. 1996. URL: https://arxiv.org/abs/alg-geom/9602006.

[5] Karen E. Smith et al. *An Invitation to Algebraic Geometry*. Springer, 2000.